

### The problem

43. Equiprobable symbols  $x \in \{-2, 2\}$  are sent over the complex AWGN channel with noise variance  $\sigma^2 = 1$ . A detector decides  $\hat{x} = 2$  if  $\text{Re}(y) > 0$  and  $\hat{x} = -2$  if  $\text{Re}(y) < 0$ . What is its error probability? Recall that  $\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-t^2} dt$  and that  $\text{erf}(u) = 1 - \text{erfc}(u)$ .
- (a)  $\frac{1}{2} \text{erfc}(2) *$
  - (b)  $\frac{1}{2} \text{erfc}(-2)$
  - (c)  $\frac{1}{2} \text{erf}(2)$
  - (d)  $\frac{1}{2} \text{erf}(-2)$

### Explanation samples

The four answers below are written at  $L = 3$  and differ only in their mathematical content.

$D = 3, L = 3$

The channel law is  $W(y|x) = \frac{1}{\pi} e^{-|y-x|^2}$ , so for equiprobable symbols the maximum-likelihood detector chooses the symbol nearest to  $y$ . Both symbols are real, hence only  $\text{Re}(y)$  matters and the boundary between the two decision regions is the midpoint  $\text{Re}(y) = 0$ . The detector of the statement is therefore the ML detector.

Suppose that  $x = 2$  is sent. The real part of the observation is  $\text{Re}(Y) = 2 + \text{Re}(Z)$ , where  $\text{Re}(Z)$  is a real Gaussian variable with zero mean and variance  $\frac{\sigma^2}{2} = \frac{1}{2}$ , since the noise variance is shared equally between the real and the imaginary parts. An error occurs when  $\text{Re}(Y) < 0$ , with probability

$$\begin{aligned} \Pr(\text{Re}(Y) < 0 \mid x = 2) &= \int_{-\infty}^0 \frac{1}{\sqrt{\pi}} e^{-(u-2)^2} du \\ &= \frac{1}{\sqrt{\pi}} \int_2^\infty e^{-t^2} dt \\ &= \frac{1}{2} \text{erfc}(2), \end{aligned}$$

where the change of variable  $t = 2 - u$  maps  $(-\infty, 0)$  onto  $(2, \infty)$ . By symmetry, the error probability given  $x = -2$  is the same, so the average error probability is also  $\frac{1}{2} \text{erfc}(2)$ , about  $2.3 \cdot 10^{-3}$ . The answer is (a).

*Marking:* Complete and correct: why the detector is ML, why the variance is  $\frac{1}{2}$ , and why the average equals the conditional value. The equations are part of the sentences and the result is interpreted.  $D = 3, L = 3$ .

$D = 2, L = 3$

The detector compares the real part of the observation with the midpoint between the two symbols, so it decides the nearest symbol. Suppose that  $x = 2$  is sent. The real part of the observation is  $\text{Re}(Y) = 2 + \text{Re}(Z)$ , and the real part of the noise is a Gaussian variable with zero mean and variance  $\frac{1}{2}$ . An error occurs when  $\text{Re}(Y) < 0$ , with probability

$$\begin{aligned} \Pr(\text{Re}(Y) < 0 \mid x = 2) &= \int_{-\infty}^0 \frac{1}{\sqrt{\pi}} e^{-(u-2)^2} du \\ &= \frac{1}{\sqrt{\pi}} \int_2^\infty e^{-t^2} dt \\ &= \frac{1}{2} \text{erfc}(2), \end{aligned}$$

with the change of variable  $t = 2 - u$ . The error probability of the detector is therefore  $\frac{1}{2} \text{erfc}(2)$ , which is option (a).

*Marking:* Mostly complete and correct. The variance  $\frac{1}{2}$  is stated without saying that the noise variance is split between the two components, and the error given  $x = 2$  is taken as the error of the detector without the symmetry argument. Nothing is wrong, something is missing.  $D = 2$ .

$D = 1, L = 3$

The detector decides the symbol nearest to the real part of the observation, so an error occurs when the noise moves the observation to the other side of zero. Suppose that  $x = 2$  is sent. The real part of the observation is  $\text{Re}(Y) = 2 + \text{Re}(Z)$ , where the noise is Gaussian with zero mean and variance  $\sigma^2 = 1$ . The error probability is the probability that this variable is negative,

$$\begin{aligned} \Pr(\text{Re}(Y) < 0 \mid x = 2) &= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-(u-2)^2/2} du \\ &= \frac{1}{2} \text{erfc}\left(\frac{2}{\sqrt{2}}\right) = \frac{1}{2} \text{erfc}(\sqrt{2}). \end{aligned}$$

This value does not appear among the options, but it is small and the only small option is (a), so the error probability is  $\frac{1}{2} \text{erfc}(2)$ .

*Marking:* Significant error with some correct reasoning. The error event and the Gaussian tail are right, but the real part of the noise gets the full variance 1 instead of  $\frac{1}{2}$ , and the answer is then chosen by elimination rather than corrected.  $D = 1$ .

$D = 0, L = 3$

The symbols are far apart compared with the noise, since their distance is 4 and the noise variance is 1, so the error probability must be small. Option (b) is larger than  $\frac{1}{2}$ , because  $\text{erfc}(-2)$  is close to 2. Option (c) is close to  $\frac{1}{2}$ , because  $\text{erf}(2)$  is close to 1, and a detector that errs half of the time is useless. Option (d) is negative, which is impossible for a probability. The only small and positive option is (a), so the error probability is  $\frac{1}{2} \text{erfc}(2)$ .

*Marking:* No explanation. The option is right, so  $Q = +1$ , but nothing derives the error probability: no decision rule, no distribution of the observation, no tail integral. Discarding the other options is not an argument for this one.  $D = 0$ .

## Use of language samples

The three answers below contain the same mathematics as the first answer above, so they receive  $D = 3$  as well. Only the writing changes.

$D = 3, L = 2$

Detector:  $\text{Re}(y) > 0 \Rightarrow \hat{x} = 2, \text{Re}(y) < 0 \Rightarrow \hat{x} = -2$ . It is the ML detector, since  $W(y|x) = \frac{1}{\pi} e^{-|y-x|^2}$  is largest for the nearest symbol and the symbols are real (only  $\text{Re}(y)$  matters).

$x = 2$  sent  $\Rightarrow \text{Re}(Y) = 2 + \text{Re}(Z)$ , with  $\text{Re}(Z)$  Gaussian, mean 0, variance  $\frac{\sigma^2}{2} = \frac{1}{2}$  (noise variance 1 split in two components).

Error if  $\text{Re}(Y) < 0$ :

$$\begin{aligned} P &= \int_{-\infty}^0 \frac{1}{\sqrt{\pi}} e^{-(u-2)^2} du \\ &= \frac{1}{\sqrt{\pi}} \int_2^{\infty} e^{-t^2} dt \\ &= \frac{1}{2} \text{erfc}(2) \quad (t = 2 - u). \end{aligned}$$

Same for  $x = -2$  by symmetry  $\Rightarrow P_e = \frac{1}{2} \text{erfc}(2) \Rightarrow$  (a).

*Marking:* Descriptive and schematic. Every step is present, but the answer is a list joined by arrows rather than an argument, and the result is not interpreted. Some sentences remain.  $L = 2$ .

$D = 3, L = 1$

ML (nearest symbol,  $W(y|x) = \frac{1}{\pi} e^{-|y-x|^2}$ ). Symbols real  $\Rightarrow \text{Re}(y)$ .

$x = 2$ :  $\text{Re}(Y) = 2 + \text{Re}(Z)$ ,  $\text{Re}(Z) \sim \mathcal{N}(0, \sigma^2/2) = \mathcal{N}(0, \frac{1}{2})$ .

$\text{Re}(Y) < 0$ :

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{\sqrt{\pi}} e^{-(u-2)^2} du &= \frac{1}{\sqrt{\pi}} \int_2^{\infty} e^{-t^2} dt \\ &= \frac{1}{2} \text{erfc}(2). \end{aligned}$$

$x = -2$  same (sym.).  $P_e = \frac{1}{2} \text{erfc}(2)$ . (a)

*Marking:* Almost no sentences. The words are labels attached to formulas, and the reader has to reconstruct the reasoning that links them.  $L = 1$ .

$D = 3, L = 0$

$$\begin{aligned} W(y|x) &= \frac{1}{\pi} e^{-|y-x|^2}, \quad \text{ML}, \quad \text{Re} \\ \text{Re}(Z) &\sim \mathcal{N}(0, \frac{1}{2}) \\ \int_{-\infty}^0 \frac{1}{\sqrt{\pi}} e^{-(u-2)^2} du &= \frac{1}{\sqrt{\pi}} \int_2^{\infty} e^{-t^2} dt = \frac{1}{2} \text{erfc}(2) \\ P_e(-2) &= P_e(2) \\ P_e &= \frac{1}{2} \text{erfc}(2) \quad (\text{a}) \end{aligned}$$

*Marking:* No writing at all. The formulas contain the whole argument, with ML and Re as the only words. There is nothing to mark as language.  $L = 0$ .